INDIAN STATISTICAL INSTITUTE
BANGALORE CENTRE

SEMESTRAL EXAMINATION
M. MATH, II YEAR, II SEMESTER, 2014-15
ERGODIC THEORY

Time limit: 2 1/2 hours Maximum marks: 100*

*The six questions below carry a total of 110 marks. Answer as many questions as you can.

1.

a) Let T be a measurable map on a probability space $(\Omega, \mathcal{F}, P)$ such that $P \circ T^{-1} << P$. Suppose $f \circ T \in L^1$ for every f in L^1. Show that $f \to f \circ T$ is a bounded linear map on L^1.

b) In addition to the hypothesis of a) assume that $\{\frac{1}{n}\sum_{k=0}^{n} f \circ T^k\}$ converges in the norm of L^1. Prove that there exists a finite constant C such that $\frac{1}{n}\sum_{k=0}^{n} P(T^{-k}(E)) \leq CP(E)$ for all $E \in \mathcal{F}$. [15+15]

2. Let T be an ergodic measure preserving transformation on a probability space $(\Omega, \mathcal{F}, P)$ and $A, B \in \mathcal{F}$ with $P(A) > 0$ and $P(B) > 0$. Show that $P(A \cap T^n B) > 0$ for some integer n. [15]

3. Let $\Omega = [0, 1), \mathcal{F}$ the Borel sigma field and P the Lebesgue measure. Let $Tx = 2x \operatorname{mod}(1)$ and $\mathcal{A} = \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$. Compute $h(\mathcal{A}, T)$. [20]

4. Recall the construction of Kakutani Tower in Rokhlin's Lemma: $\epsilon > 0$ and $N \in \mathbb{N}$ are given and we choose B with $0 < P(B) < \frac{\epsilon}{N}$; $B_j = \{\omega \in B : T\omega \notin B, T^2\omega \notin B, ..., T^{j-1}\omega \notin B, T^j\omega \in B\}$ and the sets in the tower are $T^i(B_j) : 0 \leq i \leq j-1, j = 1, 2, ...$ Let E be the union of all the sets in this tower. Prove that $P(E) = 1$. [15]

5.

a) Let T be a topologically transitive homeomorphism of S^1. Show that all continuous invariant functions are constants.

b) Let T be a minimal homeomorphism of S^1. Show that $T^n z = z($ where $z \in S^1$ and $n \in \mathbb{Z}) \Rightarrow n = 0$. [5+5]

6. Let $T : S^1 \to S^1$ be defined by $Tz = az$ where a is a fixed point of S^1. Show that T is ergodic if and only if a is not a root of unity. [20]